**SPECIAL TRIGONOMETRIC FUNCTIONS**

**4.1. Standard Integrals.**

**(A) .**

**Proof.**

=

( on multiplying numerator and denominator by sec2 )

=

Since, numerator is the diff. coeff. Of denominator.

**(B)**

=

Proof.

**=**

=

=

**INTEGRAL CALCULUS**

**Note.** Alternative Methods :

Since the numerator is the derivative of the denominator.

= = , where z = sin x

=

, where z = tan ½x

=

It should be noted that the different forms in which the integrals of cosec x and of sec x are obtained by different methods can be easily shown to be identical by elementary trigonometry.

Thus,

* Query successful

= log | tan ½x | ; etc.

**SPECIAL TRIGONOMETRIC FUNCTIONS**

**4.2**

The given integral

=

=

( on multiplying the numerator and denominator by sec2 )

**Case I. a > b.**

Put

The given integral now becomes

= [ see (A), Art. 2.3.]

=

i.e., =

**Case II. a<b.**

**Put**  ...

**INTEGRAL CALCULUS**

As before, the required integral becomes

[See (C), Art. 2.3]

= (1/√(b² - a²)) log |(√(b+a) + √(b-a) tan ½x) / (√(b+a) - √(b-a) tan ½x)|

**Note 1**. Here it is assumed that a > 0, b > 0; if a < 0, b > 0 or, a > 0, b < 0, or, a < 0, b < 0, then the integral can be evaluated exactly in the same way.

**Note 2**. (i) If b=a, the integrand reduces to sec² , the integral of which is tan ½x.

(ii) If b = -a, the integrand reduces to cosec² , the integral of which is cot ½x.

**Note 3.** By an exactly similar process, the integral or more generally can be evaluated by breaking sin x and cos x in terms of ½x and then multiplying the numerator and the denominator of the integrand by sec² ½x and substituting z for tan ½x. This is illustrated in Examples 3 and 4 of Art. 4⋅8 below.

In fact any rational function of sin x, cos x can be easily integrated by expressing sin x and cos x in terms of tan ½x, i.e., by writing

and cos x =

And then putting ½ x = z.

Similar integrals involving *hyperbolic functions* can be evaluated by an exactly similar process.

**SPECIAL TRIGONOMETRIC FUNCTIONS**

**4.3 Positive integral powers of sine and cosine.**

(A) Odd positive index.

Any odd positive power of a sine and cosine can be integrated immediately by substituting cos x = z and sin x = z respectively as shown below.

Ex. (i)

= [Putting z for cos x]

= .

Ex. (ii)

= [Putting z for sin x]

=

=

(B) Even positive index.

In order to integrate any even positive power of sine and cosine, we should first express it in terms of multiple angles by means of trigonometry and then integrate it.

**Note 1.** It should be noted that when the index is large, it would be more convenient to express the powers of sines or cosines of angles in terms of multiple angles by the use of De Moivre's Theorem, as shown below.

**Ex. (iv) Integrate .**

Let

then, .

...

...

=

=

=

...

...

=

=

**Note 2**. When the index is an odd positive integer, then also we can first express the function in terms of multiple angles and then integrate it; but in this case, it is better to adopt the method shown above in (A).

Thus, .

**4.4. Products of positive integral powers of sine and cosine.**

Any product of the form admits of immediate integration as in Sec A, Art. 4⋅3, whenever *either p or q* is a positive odd integer, whatever the other may be. But when *both p and q are positive even indices*, we may first express the function as the sum of a series of sines or cosines of multiples of *x* as in Sec. B, Art. 4⋅3, and then integrate it.

**Ex. (i) Integrate**

I =

=

= , [ putting z=sin x ]

=

=

=

**Note.** The expression x also admits of immediate integration in terms of or if be a negative even integer, whatever p and q may be. In this case, the best substitution is or . For other cases of x, a reduction formula is generally required. See 8⋅14–8⋅17.

Ex. (iii) Integrate .

Here, = 2 - 6 = - 4; ∴ put tan x = z, then sec²x dx = dz.

Now, I=

=

=

**4.5. Integral powers of tangent and cotangent.**

Any integral powers of tangent and cotangent can be readily integrated. Thus,

=

= ∫ cot²x cosec²x dx - ∫ cot²x dx

= - ∫ cot²x d (cot x) - ∫ (cosec²x - 1) dx

= -⅓ cot³x + cot x + x.

**4.6 Positive integral powers of secant and cosecant.**

(A) *Even positive index.*

Even positive powers of secant or cosecant admit of immediate integration in terms of tan *x* or cot *x*. Thus,

=

=

=

=

=

(B) *Odd positive index.*

Odd positive powers of secant and cosecant are to be integrated by the application of the rule of integration by parts.

... transposing ∫ sec³x dx to the left side, writing the value of ∫ sec x dx, and dividing by 2, we get

1. ∫ = ∫ sec³x sec²x dx

= sec³x tan x - ∫ 3 sec³x tan²x dx

= sec³x tan x - 3 ∫ sec³x (sec²x-1) dx

= sec³x tan x + 3 ∫ sec³x dx - 3 ∫.

Now, transposing 3 ∫ dx and writing the value of ∫ sec³x dx, we get ultimately,

1. ∫ cosec³x dx = ∫ cosec x cosec²x dx

= -cosec x cot x - ∫ cosec x cot²x dx

= -cosec x cot x - ∫ cosec x (cosec²x-1) dx

= -cosec x cot x + ∫ cosec x dx - ∫ cosec³x dx.

.: transposing ∫ cosec³x dx and writing the value of ∫ cosec x dx, ∫ cosec³x dx = -½ cosec x cot x + ½ log tan ½x.

**4.7. Hyperbolic Functions.**

(i) ∫ = ∫ ½(eˣ - e⁻ˣ) dx = ½(eˣ + e⁻ˣ) = cosh x.

(ii) ∫ = ∫ ½(eˣ + e⁻ˣ) dx = ½(eˣ - e⁻ˣ) = sinh x.

(iii) ∫ tanh x dx .

(iv) ∫ = log |sinh x|.

(v) ∫ cosech x dx = =

= 2 ∫

=

= = log |tanh ½x|.

[on dividing the numerator and denominator by e^(½x)]

(vi) =

=

=*.*

Otherwise:

(vii)

(viii) ∫.

(ix) ∫*.*

(x) ∫

**4.8 Illustrative Examples.**

**Ex. 1** Integrate .

Put , , *then .*

Here r = ​ and θ =

...

=

=

=

Note. Since, as above

Ex. 2.

Multiplying the numerator and denominator by , this  
  
=   
  
=

= =

=

=

Ex 5.

Now comparing coefficient’s of on and on both sides, we got and when ,

**Note:** Generally can be treated in the same way.

Ex. 6. Integrate .

Ex. 7. Integrate .

Ex. 8. Integrate

( on multiplying the numerator and denominator by .)  
Put ; then .  
.

**EXAMPLES IV**

Integrate with respect to the following functions :-  
1.  
(i) .  
[ C. P. 1929 ]  
(ii) .  
(iii) .  
(iv) .  
(v) .  
(vi) .  
(vii)   
(viii) .  
(ix) .  
(x) .  
(xi) .  
(xii) .  
(xiii) .  
2.  
(i)   
(ii)   
(iii)   
(iv)   
(v)   
(vi)

Evaluate the following integrals :-  
3.  
(i) .  
(ii) .

(iii) .  
(iv) .  
(v) .  
(vi) .  
4. (i) .  
(ii)   
5. (i) .  
(ii) .  
6.   
7. .  
8. (i) .  
(ii) .  
9. (i)   
(ii) .  
10. .  
11. .  
12. (i) .  
(ii) .  
13. (i) .  
(ii) .  
(iii) .  
(iv) .  
[Put in the numerator of (i) and (ii).]  
14.  
(i) .  
(ii) .  
[ Put tan in (i) and (ii).]  
15.  
(i) .  
(ii) .  
16. (i) .  
(ii) .  
17.  
(i) .  
(ii) .  
[ (ii) Write .]  
18. (i) .  
(ii) .  
19. .  
20.  
(i) .  
(ii) .  
21.  
(i) .  
(ii) . [ ( ( ) Numerator ]  
(iii) .  
22. .  
23.   
24.   
[C. P. 1933 ]  
25.  
(i)   
(ii)   
(iii)   
(uv)   
26. (i)   
(ii)   
27.  
(i)   
(ii)   
28.   
29.

30.   
31.   
32. (i)   
(ii)   
33.   
34.   
35. (i) .  
(ii) .  
36. (i) .  
(ii) .  
[(i) Put and note .]  
37. .  
38.

**ANSWERS**

1.

(i)   
(ii)   
(iii)   
(iv)   
(v)   
(vi)   
(vii)   
(viii)   
(ix)   
(x)   
(xi)   
(xii)   
(xiii)   
2.  
(i)   
(ii)   
(iii)   
(iv)

(v)   
(vi)   
3. (i)   
(ii)   
(iii)   
(iv)   
(v)   
(vi)   
4. (i)   
(ii)   
5. (i) & (ii)   
6.   
7.   
8. (i)   
(ii)   
9. (i)   
(ii)   
10.   
11.   
12. (i)   
(ii)   
13. (i)   
(ii)   
(iii)   
(iv)   
14. (i)   
(ii)   
15. (i)   
(ii)   
16.  
(i)   
(ii)   
17.  
(i)   
(ii)   
18. (i) .  
(ii)   
19.   
20. (i)   
(ii)   
21.  
(i)   
(ii)   
(iii)   
22.   
23.

24. , if ;

25.

(i)   
(ii)   
(iii)   
(iv)   
26. (i)   
(ii)   
27. (i)   
(ii)   
28. , if ;  
 if   
29. , if , if .  
30.   
31.

32. (i)   
(ii)   
33.   
34.   
35. (i)   
(ii) .  
36.   
37.   
38.